Hierarchical pre-segmentation without prior knowledge

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Abstract

A new method to pre-segment images by means of a hierarchical description is proposed. This description is obtained from an investigation of the deep structure of a scale space image – the input image and the Gaussian filtered ones simultaneously. We concentrate on scale space critical points – points with vanishing gradient with respect to both spatial and scale direction. We show that these points are always saddle points. They turn out to be extremely useful, since the iso-intensity manifolds through these points provide a scale space hierarchy tree and induce a segmentation without a priori knowledge. Moreover, together with the so-called catastrophe points, these scale space saddles form the critical points of the parameterised critical curves – the curves along which the spatial saddle points move in scale space. Experimental results with respect to the hierarchy and segmentation are given, based on an artificial image and a simulated MRI.

1 Introduction

One way to understand the structure of an image is to embed it in a one-parameter family. If a scale-parametrised Gaussian filter is applied, the parameter can be regarded as the “scale” or the “resolution” at which the image is observed. The resulting structure has become known as linear, or Gaussian, scale space. Main advantage is that this set of filters enables one to take derivatives of a discrete image. More detailed literature can be found in e.g. [2, 14, 16, 20].

In their original accounts both Koenderink [8] and Witkin [26] proposed to investigate the “deep structure” of an image, i.e. structure at all levels of resolution simultaneously. Encouraged by the results in specific image analysis applications, an increasing interest has recently emerged trying to establish a generic underpinning of deep structure. Results from this may serve as a basis for a diversity of multiresolution schemes. Such bottom-up approaches often rely on catastrophe theory [4, 22], which is now fairly well-established in the context of the scale-space paradigm. The application of catastrophe theory in Gaussian scale space has been studied e.g. by Damon [1]—probably the most comprehensive account on the subject—as well as by others [3, 5, 6, 7, 9, 12, 13, 14, 17, 18].

The first stage in using the deep structure is to link image properties of two subsequent resolution scales. Although this may seem obvious, it is a non-trivial task in a discrete scale space. For example, if extrema at different scales correspond to an extremum at the input image, they should be linked. However, extrema may be annihilated or created. Tracking over scale therefore needs a cautious approach. Koenderink [8] mentioned a possible linking strategy using the properties of the Gaussian scale space. However, only a few heuristic attempts have been made to build such multiscale datastructures, e.g. by Vincken et al. [23]. Simmons et al. [19] used the idea of Koenderink’s scheme for building a so-called extremum stack. However, they ignored the generic possibility of creations and only used the annihilation intensity. Their work was an extension of the results by Lifshitz and Pizer [12], who implemented Koenderink’s scheme, mainly focusing on heuristics and the performance of the algorithm. At the annihilation of a minimum and a saddle point they noticed that the saddle point decreased in intensity, but passing the zero-crossing of the Laplacean, close to the annihilation, started to increase again. In response to their research Koenderink [9] showed that this happens generically for 2D saddles. Moreover, these saddle points with zero-Laplacean are saddle points in scale space.

The idea of tracing critical points and using the location where they vanish as input for a hierarchy tree was also proposed and implemented by Zhao and Iijima [27], cited in [25], and by Wada and Sato [24].

Special behaviour of critical curves at scale space saddles has been mentioned in literature by few other authors. Griffin and Colchester [5] pointed out that at a catastrophe the saddle and the extremum necessarily have the same sign of the Laplacean and distinguished between ridge and
trough saddles. Therefore saddles change from ridge to trough or vice versa. Lindeberg [13, 14] investigated the locations of Laplacean zero-crossings in combination with the (annihilation of) critical points and concluded that “in two and higher dimensions there is no absolute relation between locations of the Laplacean zero-crossing curves and the local extrema of a signal”.

The aim of this paper is to combine knowledge from catastrophe theory, properties of scale space, particularly with respect to the scale space saddles, and the multi-scale linking strategy as suggested by Koenderink. In section 2 we explain basic principles and show that scale space saddles are the key to explore the deep structure of scale space images. They give rise to the unambiguous multi-scale hierarchy describing the image presented in section 2.5. Images in one dimension fundamentally differ from those in higher dimensions, since only in 1D images the scale space saddles coincide with the catastrophe points. Therefore both cases are discussed separately. The results lead to a non-heuristic hierarchical multi-scale data structure and a segmentation of images without any a priori knowledge. Section 3 shows results on simple images and a 2D synthetic MRI. Main conclusions and results are given in section 4.

2 Theory

2.1 Deep Structure in Gaussian Scale Space

Given an arbitrary n-dimensional image \( L(x) \), we denote its Gaussian scale space image by \( L(x; t) \) satisfying \( \partial_t L = \Delta L \). Spatial critical points (extrema, saddles) of \( L(x; t) \) at certain scale \( t_0 \) are defined as the points where \( \nabla L(x; t_0) = 0 \). The behaviour of spatial critical points as the (scale) parameter changes is described by catastrophe theory. As the parameter continuously changes, the critical points move along critical curves, defined as a one dimensional manifold in scale space on which \( \nabla L(x; t) = 0 \).

If the determinant of the Hessian does not become zero, these critical points are called Morse critical points. In a typical image these points are extrema (minima and maxima) or saddles. The Morse lemma states that the topology of a neighbourhood of a Morse critical point can essentially be described by a second order polynomial. At isolated points on a critical curve the determinant of the Hessian may become zero. These points are called non-Morse points. Neighbourhoods of such points need a third or higher order polynomial, as described by Thom’s theorem [22].

2.2 Scale Space Critical Points

Scale space critical points of \( L(x; t) \) are defined as the points with zero gradient and zero Laplacean: \( \nabla L(x; t) = 0 \wedge \Delta L(x; t) = 0 \), since \( \partial_t L(x; t) \) by definition. The type of these critical points is determined by the eigenvalues of the matrix of second order derivatives, \( H \).

We call this matrix the extended Hessian:

\[
H = \begin{pmatrix} H & \Delta L \\ \Delta L^T & \Delta L \end{pmatrix}. \tag{1}
\]

Here \( H \) is the spatial Hessian defined by \( H_{i,j} = L_{i,j} \), all evaluated at the location of the critical point of interest. Points are maxima (minima) if all eigenvalues are all negative (positive). If at least two eigenvalues have a different
sign, the point is a saddle. Since $H$ is symmetric, all eigenvalues are real.

**Theorem 1** The matrix $H$ has both positive and negative eigenvalues if $\Delta L = 0$.

**Proof 1** Let the point $(x_0, t_0)$ be a critical point of the function $L(x; t)$. Then $(x_0, t_0)$ is also a critical point of the function $L(x; t_0)$ at scale $t_0$. If $(x_0, t_0)$ is an extremum of $L(x; t)$, it is also an extremum of $L(x; t_0)$. But then the extremum principle states that the Laplacean is non-zero. So $(x_0, t_0)$ is a saddle point. □

As a consequence, critical points in scale space are always saddle points. These scale space saddle points form a subset of the spatial saddles, since critical points with vanishing Laplacean in spatial sense are always saddle points.

This notion extends the idea of non-creation of local (spatial) extrema, valid only in the one dimensional case, but sometimes erroneously extended to higher dimensions. It is known from catastrophe theory that on a finite scale interval each branch of the critical curve is bounded with respect to scale: at some scale the critical points annihilate. Critical points are present from the initial scale or they are created at a certain (catastrophe) point in scale space. If the scale is taken coarse enough only one extremum remains. The conditions for this to be true are discussed in [15]. Then there exists one critical curve bounded by the coarsest scale. Apart from catastrophe points a second type of points exhibits special behaviour, viz. scale space saddles.

On critical curves the intensities of the critical points is well-defined. The intensity of extrema is damped continuously in scale space. Each minimum (maximum) therefore increases (decreases) monotonically towards its annihilation point. At certain spatial and scale distance from the annihilation, the intensity of corresponding saddle will generically tend to move towards the intensity of extremum, i.e. it decreases (increases) to the intensity of minimum (maximum). So the signs of the Laplacean of both critical points at that scale will be opposite. At the catastrophe point, however, they necessarily have the same sign and both points approach the intensity of the annihilation while decreasing (increasing) in value.

Therefore, at saddle-branch of the critical curves, the saddle will generically pass a point at which the Laplacean equals zero: a scale space saddle. Since the sign of the Laplacean changes passing the scale space saddle, the intensity changes from increasing to decreasing or vice versa.

A parametrisation of a critical curve leads to a 1D-function of the intensity of the critical points. The extrema of this function have the following properties:

**Theorem 2** Let $(x(s); t(s))$ be a parametrisation of $(x; t)$, such that $\nabla L(x(s); t(s)) = 0$, i.e. $(x(s); t(s))$ defines a critical curve. Then $L(x(s); t(s))$ has its extremum at the scale space saddle(s) and the catastrophe point(s). For 1D images (signals) the parametrisation has a point of inflection.

**Proof 2** The total differentiation of $L(x(s); t(s))$ with respect to $s$ is defined by

$$\frac{dL(x(s); t(s))}{ds} = \nabla L \cdot \frac{dx}{ds} + \Delta L \cdot \frac{dt}{ds}.$$ (2)

Since $\nabla L = 0$, the critical points of Eq. (2) are given by $\Delta L \cdot \frac{dt}{ds} = 0$. The scale space saddles are defined as the points where $\Delta L = 0$, whereas the catastrophes take place at the location where the saddle and the extremum ‘meet’ in scale space, i.e. where the parametrisation of scale has its local extremum. The critical points of $L(x(s); t(s))$ are extrema, since both the Laplacean and the catastrophe point are non-degenerate and do not coincide for n-D images, $n > 1$. For 1D images they do, so the point is a point of inflection. □
Although this results holds for any parametrisation of the critical curves, in practice the intensities of critical points are obtained at the calculated scales of the scale space. In other words, they are measured as a function of scale. Then \( t = s \), so \( \frac{dt}{ds} = 1 \) and \( L(x(t); t) \) is obtained as the union of its branches. Each branch is defined on a closed interval \( s_1 \leq s \leq s_2 \), where \( s_1 \) is either the initial or the creation scale, and \( s_2 \) is the annihilation scale of the spatial critical point.

2.4 The Structure of Iso-Intensity Manifolds

Each extremum is encircled by iso-intensity manifolds occurring in four distinct shapes.

- At most intensities ‘before’ the annihilation the manifolds around the extremum are dome-shaped encapsulating a bounded region, see Figure 1a.
- At scale space saddles the manifold is dome-shaped around the extremum belonging to the saddle point. At the saddle point it touches another manifold with the same intensity, see Figure 1b.
- At intensities between the scale space saddle and the annihilation point the manifold around the extremum belongs to the saddle point. At most intensities ‘before’ the annihilation the manifold transforms from dome-shaped to horseshoe, see Figure 1c.
- At the annihilation intensity the manifold has a horseshoe shape, as known from catastrophe theory, see Figure 1d.

Figure 1. The four distinct appearances of 2D Iso-intensity surfaces in scale space (see text for discussion).

As a dual expression it follows that each extremum forms the top of an iso-intensity dome in scale space, until its intensity equals that of the related scale space saddle. Then the dome transforms to a horseshoe shape at the annihilation. In case of a minimum (maximum) there are only pure domes at intensities smaller (larger) than the intensity of the scale space saddle.

2.5 Scale Space Hierarchy

Since each extremum encapsulates a series of domes from the initial scale to a scale space saddle, the intensities of the collection of these domes define a segment. The boundaries of various segments follow directly from the intensities of the scale space saddles. A natural hierarchy is obtained as scale space segments are defined by the regions encapsulated by the iso-intensity manifolds through the scale space saddles. This hierarchy avoids the problems of a straightforward segmentation of an image based on the intensities of the saddle points. Although saddles have different intensities in the initial image (since they are Morse-saddles), at some scales intensities of saddles are equal, see e.g. Lindeberg [14]. Then, for example, a saddle isophote contains another saddle and encircles three extrema. In scale space, however, the scale space saddles generically have different intensities.

The hierarchy tree contains as nodes the scale space saddles and their intensities. The branches are formed by the segments, defined by the collection of internal domes, bounded by the iso-intensity manifold through the scale space saddle. So one branch represents the set of domes of the corresponding extremum. The scale space saddles are ordered by scale. Segments in the tree can be joined if they have a scale space saddle in common. Subtrees contain parts of the image and can be selected or deselected. To obtain a simple segmentation, only the part of the tree with large scales can be regarded. See section 3 for applications.

The scale space hierarchy is uniquely found by the algorithm based on the next six steps:

1 Scale Space: Input is an image of arbitrary size and dimension. Only for the sake of illustration we consider the one and two dimensional cases. Images of higher dimension are comparable to the two dimensional ones, albeit they allow saddle-saddle pairs. A scale space image is obtained by convolving the input image with a normalised Gaussian filter of variable size. The intermediate levels are logarithmically sampled, see e.g. [2, 8, 10, 14].

2 Extremum and Saddle Stacks: Each level in scale space is a blurred image. Its critical points can be calculated by various methods, e.g. zero-crossings of the derivatives, winding-numbers, or neighbourhood-relations.

3 Extremum and Saddle Paths: Since critical points can be annihilated and created, they may undergo, apart form movement in scale direction, also purely spatial drift. This movement can be calculated accurately by means of derivatives up to third order, see e.g. [3, 11]. The outcome of this procedure are two stacks each containing doubly linked lists. The head of each list corresponds with the creation of the critical point (or the initial scale), its tail with the annihilation.

4 Connected Critical Paths: Since the annihilation of an extremum involves a saddle, each tail of an extremum list at a certain scale \( i \) corresponds to a tail of some saddle list at the same scale \( i \). At catastrophes the movement of a critical point can be accurately predicted, see [3, 11]. This results
in chains of extremum-saddle pairs, viz. critical curves.

5 Scale Space Saddles: Scale space saddles have the property that they are the local extrema of the parametrised intensity-curve, obtained by taking the intensity along the saddle branches as function of scale, as argued in section 2. Saddle lists can have zero or multiple extrema with respect to intensity. If no extrema are found then the Laplaceans of the extremum and the saddle have either the same or the opposite sign at all scales. The former signals that the scale space saddle is absent. To identify a segment with the extremum, the intensity of the saddle in the first image of the scale space stack can be taken. The latter case represents a scale space saddle located closer to the catastrophe point than is measured. The saddle at the coarsest scale is assigned as scale space saddle. If multiple scale space saddles are found within one saddle list, the one at coarsest scale is chosen. Since each extremum list is linked to a saddle list, each extremum is linked to a scale space saddle. Equivalently, the iso-intensity manifold through the scale space saddle encapsulates the corresponding extremum.

6 Hierarchical Tree: The scale space saddles are sorted from coarse to fine according to scale at which the extremum saddle pair annihilates. Each scale space saddle defines an iso-intensity manifold around an extremum: the part of the image encapsulated by this manifold is a segment of the image at that scale. Segments may have sub-segments, defined by scale space saddles within the segment. At the coarsest scale only one extremum remains. Since it has no corresponding saddle branch containing a scale space saddle, it does not have an a priori critical dome. These domes, however, are defined as the iso-intensity manifolds through an extremum at a pre-defined scale, viz. at which a scale space saddle occurs. Therefore the iso-intensity manifold of the last extremum can be chosen having the intensity of the extremum at the coarsest scale. Since the heat equation is energy preserving, it is known that the input image converges to an image of constant value equaling the average value of the input image. Consequently the value of the iso-intensity manifold of the remaining extremum can be set to this value.

2.6 Segmentation

A natural segmentation, or rather "pre-segmentation", of scale space is thus obtained by the iso-intensity manifolds of the scale space saddles with their corresponding extrema. Consequently a spatial segmentation of the image at any scale is found by the intersection of the scale space segmentation and this fixed scale; A full (partial) segmentation of the initial image is found by taking into account the intensities of all (a proper subset of all) scale space saddles. At a partial segmentation each selection of scale space saddles define segments with a certain grey-level histogram.

If knowledge of the grey-level distribution of the image is present, this leaves room for a semantical choice.

3 Applications

To show the results of the scale space saddle hierarchy and pre-segmentation in 2D we took an 81x81 artificial image, made by the combination of four maxima and one (induced) minimum, see Figure 2a. The simplicity of this image enables a quantitative check of the outcome.

Since the image at (very) large scale contains only one blob, 4 extrema must be annihilated. To obtain the scale space hierarchy firstly a scale space consisting of 113 levels was built. Levels were calculated at scales $e^{t/10}$, $t = 2, \ldots, 114$. Secondly at each level the spatial critical points were calculated.

Thirdly the spatial critical points of subsequent scales were linked resulting in the critical paths. Figure 3 shows the locations of spatial critical points in scale space. For visualisation purposes this 81x81x113 space was reduced to a 41x41x13 volume of interest space. Dark points correspond to extrema, light points to saddle points. At three isolated scales a pair of created and directly annihilated critical points is visible. The algorithm is able to detect these points and finds the right linking.

Fourthly extrema and saddle points were pairwise grouped by means of the catastrophe points. The parametrised critical paths, viz. the intensities of the critical curves containing the branches of saddle and extremum branches, are shown in Figure 4a. The four catastrophes are visible as the end of two branches of critical points.

Fifthly the scale space saddles are derived from the saddle branches. These are shown in Figure 4b. It can be seen that the upper three saddle branches, although containing multiple local extrema with respect to the intensity, have a global maximum, viz. the scale space saddle of interest. The fourth saddle branch is monotonically increasing, just as its corresponding minimum. Therefore the intensity of the spatial saddle at the first level is chosen as value for the minimum encapsulating manifold.

Figure 2. a) Artificial 81 x 81 image build by combining four maxima and one minimum. b) Pre-segmentation of the initial image.
Finally an unambiguous hierarchy based on the catastrophe points and the scale space saddles, just as in the 1D case, can be made. The presence of 5 extrema results in 5 inner regions \( S_i, i = 1, \ldots, 5 \) and a boundary region \( S_0 \). The first region is defined by the remaining extremum. The scale space dome defined by this maximum is the iso-intensity manifold corresponding to the average intensity of the initial image. This convergence can also be seen in Figure 4a. This segment \( S_1 \) and its complement \( S_0 \) projected to the initial image are shown in Figure 2b.

To find the next segment, scale is decreased until the second extremum appears. The segment \( S_2 \) corresponding to it is located at the bottom right part of the image. The value of the iso-intensity manifold is obtained from the scale space saddle of the spatial saddle corresponding to this extremum. The other segments are found in the same way, resulting in the pre-segmentation of the image as shown in Figure 2b. Furthermore the hierarchy associated with this pre-segmentation is given by the successive annihilations in scale space, shown in Figure 5.

Having a hierarchical description tree of the image, one can disregard parts of the tree. Combined with knowledge of the image one can prune the pre-segmentation to a more useful segmentation. Figure 6a shows a 2D slide from a simulated MRI of brain tissue. This image is taken from the web-site http://www.bic.mni.mcgill.ca/brainweb.

With this MRI, also the distribution of the white matter is given as ground-truth. This image is shown in Figure 6b. Selecting the manifold obtained by the intensity of the scale space saddle of the last catastrophe in the hierarchy tree of Figure 6a, Figure 6c is obtained. For such a trivial procedure this is a remarkably good result that shows the feasibility of the general scheme.

### 4 Summary and Conclusions

We developed a method to calculate the hierarchical structure of an arbitrary input image. Consequently, this structure can be represented as a pre-segmentation. The method is based on the scale space image of the input image and the critical paths within it. The latter exist of branches of extrema and saddle points. The range of scales at which these branches exist follow from their catastrophe points in scale space. To each extremum that annihilates an iso-intensity manifold is assigned. The value of this manifold equals that of the scale space saddle located at the saddle branch annihilating with the extremum branch. This point is a critical point in scale space. The iso-intensity manifold encapsulates the extremum in scale space. The manifolds
through the extrema are nested and non-intersecting and thus form a hierarchy. Consequently, a pre-segmentation of the image without any a priori knowledge is obtained by the intersection of the image and the manifolds. The proposed algorithm has two main advantages. Firstly it has a rigorous mathematical underpinning which encourages and facilitates future improvements, and admits reproducible, predictable, and provable segmentation results. Secondly it has the potential to include semantics enabling an intelligent choice of the nodes, either by deterministic, statistic or probabilistic means. Experimental results based on artificial images and simulated MRI with respect to the hierarchy and segmentation were given and showed results that correspond to a fair degree to both the mathematical and the intuitive forecast.

References


